JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **112**, 274–278 (1985)

Iteration of Holomorphic Maps in Hilbert Spaces

GONG-NING CHEN

Department of Mathematics, Beijing Normal University, Beijing, China

Submitted by Ky Fan

Previous work (GONG-NING CHEN, *J. Math. Anal. Appl.* **98** (1984), 305–313) on iteration of holomorphic maps of C^n is continued. The purpose of this note is to extend results given in the above mentioned reference to the case of complex Hilbert spaces. Other comments are appended. © 1985 Academic Press, Inc.

1. A theorem of Wolff [9] (see also Burckel [1]) asserts that if $f: \Delta \to \Delta$, the open unit disc in the complex plane, is analytic and fixed-point-free, then there exists $u \in \partial\Delta$ such that every closed disc in Δ which is tangent to $\partial\Delta$ at u is mapped into itself by every iterate of f. Recent papers by Ky Fan [4, 5], and by the author [2] show that Wolff's theorem can be extended to analytic functions of operators, and to holomorphic maps of C^n, respectively. The present note is a continuation of [2], and deals with iteration of holomorphic maps in Hilbert spaces. The purpose here is to generalize results givin in [2] to the case of complex Hilbert spaces of any dimension. As in the previous work [2], this writing was also inspired by the elegent results of [4, 5].

Throughout this note, X and Y will be complex Hilbert spaces of any dimension, and B and B_1 the open unit balls of X and Y, respectively. A map F defined on an open connected set D in X with values in Y is said to be holomorphic if for each $x \in D$ its Fréchet derivative at x exists as a bounded complex-linear map of X into Y. By a biholomorphic map we mean a holomorphic map with a holomorphic inverse.

Fix $b \in B$, and let P_b be the orthogonal projection of X to the subspace spanned by b and set $Q_b = I - P_b$. Here I stands for the identity map on X. Denote by Φ_b the Möbius transformation

$$\Phi_b = \frac{b - P_b x - (1 - \|b\|^2)^{1/2} Q_b x}{1 - \langle x, b \rangle} \qquad (x \in \bar{B}), \tag{1}$$

where $\langle , \rangle$ denotes the inner product of X. It is well known [6, 8] that the identity

0022-247X/85 $3.00

$$1-\|\Phi_b(x)\|^2=\frac{(1-\|b\|^2)(1-\|x\|^2)}{|1-\langle x, b\rangle|^2} \tag{2}$$

holds for all $x \in \bar{B}$, and that Φ_b is a holomorphic map of B ($\bar{B}$) onto B ($\bar{B}$).

2. In this section, we extend Theorem 1 in [2] to the case of iteration for holomorphic self-maps of B of complex Hilbert spaces. Other comments are appended. We begin with a version of Lemma 1 in [2] by replacing C^n by X (with the same proof).

LEMMA 1. *Let X be a complex Hilbert space and let b be a fixed point of $\bar{B}$, $x \in B$, and $d > 0$. Then inequality*

$$\frac{|1-\langle x, b\rangle|^2}{1-\|x\|^2} \leqslant d \tag{3}$$

holds if and only if

$$\begin{aligned} T_b(x, d) \equiv \left\|P_b x-\frac{b}{d+\|b\|^2}\right\|^2+\frac{d}{d+\|b\|^2}\|Q_b x\|^2 \\ \leqslant \frac{d(d+\|b\|^2-1)}{(d+\|b\|^2)^2}. \end{aligned} \tag{4}$$

Equality occurs in (3) *if and only if it occurs in* (4). *When $b \in B$, both* (3) *and* (4) *are equivalent to*

$$\|\Phi_b(x)\|^2 \leqslant \frac{d+\|b\|^2-1}{d}, \tag{5}$$

and equalities hold simultaneously in (3), (4), *and* (5).

THEOREM 1. *Let F be a holomorphic self-map of B, and $F^{[m]}$ denote the mth iterate of F. Then there exists a $w \in \bar{B}$ such that*

$$d(w, F^{[m]}(x)) \leqslant d(w, x), \tag{6}$$

$$T_w(F^{[m]}(x), d(w, x)) \leqslant T_w(x, d(w, x))=\frac{d(w, x)\{d(w, x)+\|w\|^2-1\}}{\{d(w, x)+\|w\|^2\}^2}, \tag{7}$$

and

$$\|\Phi_w(F^{[m]}(x))\|^2 \leqslant \|\Phi_w(x)\|^2=\frac{d(w, x)+\|w\|^2-1}{d(w, x)} \tag{8}$$

hold for any $x \in B$, *and* $m = 1, 2, \ldots$, *where*

$$d(w, x) = \frac{|1 - \langle x, w\rangle|^2}{1 - \|x\|^2}, \tag{9}$$

and $T_w(y, d(w, x))$ *is defined as in* (4) *for* $y \in B$. *If, further,* $F\colon B \to B$ *is continuous in the weak topology of* X, *then we have* $F(w) = w$ *unless* $\|w\| = 1$.

Proof. Take $0 < t_k < 1$ with $\lim_{k \to \infty} t_k = 1$, and set $F_k = t_k F$. As was stated in the proof of Theorem 1 in [2], by an application of a theorem of Earle and Hamilton [3] we can choose a sequence $\{x_k\}$in B such that $F_k(x_k) = x_k$ for all k. Since $\bar{B}$ is weakly compact in X, we may assume by replacing a suitable subsequence that $\{x_k\}$ converges weakly to w, i.e., $\lim_{k \to \infty} \langle y, x_k\rangle = \langle y, w\rangle$ for any $y \in X$. Then $\|w\| \leqslant \underline{\lim}_{k \to \infty} \|x_k\| \leqslant 1$, so that $w \in \bar{B}$. From the Schwarz–Pick lemma it follows that

$$\|\Phi_{x_k}(F_k(x))\| \leqslant \|\Phi_{x_k}(x)\| \qquad \text{for any } x \in B.$$

In view of the identity (2), this is equivalent to

$$\frac{|1 - t_k\langle F(x), x_k\rangle|^2}{1 - t_k^2\|F(x)\|^2} \leqslant \frac{|1 - \langle x, x_k\rangle|^2}{1 - \|x\|^2}.$$

By letting $k \to \infty$, we derive that the case $m = 1$ *of* (6) holds for any $x \in B$.

If, further, F is weakly continuous in B, i.e., $F(x_k) \to^{w} F(x_0)$ whenever $x_k \to^{w} x_0$ and $x_k's$, $x_0 \in B$, then we have $F(w) = w$ unless $\|w\| = 1$. The rest of this proof follows now from Lemma 1 by a similar argument to that used in Theorem 1 in [2]. This completes the proof.

For finite dimensional X, the condition here on F coincides with that of Theorem 1 in [2], since the weak continuity of F is then a consequence of the condition of holomorphy. Thus, Theorem 1 here is a generalization of Theorem 1 in [2].

It is incidentally seen from the proof of Theorem 1 that if F is a holomorphic map of B into B, and is weakly continuous in $\bar{B}$, then F has a fixed point in $\bar{B}$. In particular, if F is a biholomorphic map of B onto B, it is well known [6, 8] that $F = U\Phi_b$, where U is unitary and $b = F^{-1}(0)$; therefore F is weakly continuous in $\bar{B}$ since Φ_b is. This leads to the following.

Corollary 1 (Hayden and Suffridge [6]). *Any biholomorphic map of* B *onto* B *fixes a point in* $\bar{B}$.

Like Wolff's theorem, quoted above, Theorem 1 has an appealing geometric interpretation. For any $x \in B$, let $E(w, x)$ denote the set of all $z \in X$ that satisfy

$$T_w(z, d(w, x)) \leqslant \frac{d(w, x)\{d(w, x) + \|w\|^2 - 1\}}{\{d(w, x) + \|w\|^2\}^2},$$

where $d(w, x)$ is as in (9). Then $E(w, x)$ is the closed ellipsoid with center at $w/(d(w, x) + \|w\|^2)$. Let $\partial E(w, x)$ be the boundary of $E(w, x)$. We have:

COROLLARY 2. *Suppose that $F: B \to B$ is holomorphic. Then there exists $w \in \bar{B}$ such that $F^{[m]}(E(w, x)) \subset E(w, x) \subset \bar{B}$ for all $x \in B$ and $m = 1, 2,\ldots$. If $\|w\| = 1$, $E(w, x)$ is tangent to ∂B at w for all $x \in B$.*

Proof. A little manipulation shows that $E(w, x) \subset \bar{B}$ for any $x \in B$, and if, in particular, $\|w\| = 1$, then $E(w, x)$ is tangent to ∂B at w. Then, (7) means that $x \in \partial E(w, x)$ for any $x \in B$, and $F^{[m]}(x) \in E(w, x)$ for all $x \in B$ and $m = 1, 2,\ldots$. If $y \in \partial E(w, x)$, we have by Lemma 1 that $d(w, y) = d(w, x)$, whence $F^{[m]}(y) \in E(w, y) = E(w, x)$. If y is in the interior of $E(w, x)$, then we also have $F^{[m]}(y) \in E(w, x)$ for any m, for, if not, we would have that $F^{[m]}(y) \notin E(w, x)$ for some m. This implies that there exists u belonging to $\partial E(w, x) \cap \partial E(w, y)$, so that $d(w, x) = d(w, u) = d(w, y)$, a contradiction. The corollary is proven.

As for (6) and (8) with the case of $\|w\| < 1$, both of them are equivalent to the fact that

$$\rho(w, F^{[m]}(x)) \leqslant \rho(w, x)$$

holds for all $x \in B$ and $m = 1, 2,\ldots$, where $\rho(x, y) = \tanh^{-1}\|\Phi_x(y)\|$ is called the CRF (Carathéodory–Reiffen–Finsler) metric on B (see [3, 7]).

3. The rest of this note is devoted to a generalization of Theorem 2 in [2]. Let e be a unit vector in X, and let $IMu = \operatorname{Im}\langle u, e\rangle - \|Q_e u\|^2$ for any $u \in X$. Then the corresponding upper half-plane of X is defined by

$$E = \{u \in X: IMu > 0\}.$$

It is well known [7] that E is an unbounded convex domain in X, and $\bar{E} = \{u \in X: IMu \geqslant 0\}$. As a consequence of the preceding theorem, we have:

THEOREM 2. Let G be a holomorphic self-map of E, and let $G^{[m]}$ be the mth iterate of G. Then either

$$IMG^{[m]}(u) \geqslant IMu \tag{10}$$

holds for any $u \in E$ and $m = 1, 2,\ldots$, or there exists a $v \in \bar{E}$ such that

$$h(v, G^{[m]}(u)) \leqslant h(v, u) \tag{11}$$

holds for any $u \in E$ and $m = 1, 2, \ldots$, where

$$h(v, u) = \frac{|i\{\langle u, e\rangle - \langle e, v\rangle\} + 2\langle Q_e u, v\rangle|^2}{IMu}. \tag{12}$$

If, further, $G: E \to E$ is continuous in the weak topology of X, then we have $G(v) = v$ unless $IMv = 0$.

Proof. Let ψ be the Cayley transformation defined by

$$\psi(x) = \frac{i(x+e)}{1 - \langle x, e\rangle}$$

for any $x \in X$ with $\langle x, e\rangle \neq 1$. It is well known [7] that ψ is a biholomorphic map of B onto E. Thus $F = \psi^{-1} \circ G \circ \psi$ is a holomorphic map of B into B. This proof is then completed in a manner similar to that of Theorem 2 in [2], and therefore is omitted.

Acknowledgment

I wish to thank Professor Ky Fan for his helpful comments during the preparation of this short note.

References

1. R. B. Burckel, Iterating analytic self-maps of discs, *Amer. Math. Monthly* **88** (1981), 396–407.
2. Gong-ning Chen, Iteration for holomorphic maps of the open unit ball and the generalized upper half-plane of C^n, *J. Math. Anal. Appl.* **98** (1984), 305–313.
3. C. J. Earle and R. S. Hamilton, A fixed point theorem for holomorphic mappings, *in* "Global Analysis; Proceedings, Symposium Pure Math. Vol. XVI. Berkeley, Calif., 1968," pp. 61–65, Amer. Math. Soc., Providence, R.I., 1970.
4. Ky Fan, Iteration of analytic functions of operators, *Math. Z.* **179** (1982), 293–298.
5. Ky Fan, Iteration of analytic functions of operators, II, *Linear and Multilinear Algebra* **12** (1983), 295–304.
6. T. L. Hayden and T. J. Suffridge, Biholomorphic maps in Hilbert space have a fixed point, *Pacific J. Math.* **38** (1971), 419–422.
7. L. A. Harris, Bounded symmetric homogeneous domains in infinite dimensional spaces, *in* "Lecture Notes in Math. Vol. 364" (A. Dold, Heidelberg and B. Eckmann, Zürich, Eds), pp. 13–40. Springer-Verlag, Berlin/New York/Heidelberg, 1974.
8. W. Rudin, "Function Theory in the Unit Ball of C^n," Springer-Verlag, New York, 1980.
9. J. Wolff, Sur une généralisation d'un théorème de Schwartz, *C. R. Acad. Sci. Paris* **182** (1926), 918–920; **183** (1926), 500–502.